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A Lower Bound on Independence in Terms of Degrees

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Abstract

We prove a new lower bound on the independence number of a simple connected graph in terms of its degrees.

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1 Introduction

We consider *finite*, *simple*, and *undirected graphs* G with *vertex set* V. For a graph G, we denote its *order* by n and its *size* by m, respectively. The *degree* of u in G is denoted by $d(u)$ and Δ is the *maximum degree* of G. A set of vertices $I \subseteq V$ in a graph G is *independent*, if no two vertices in I are adjacent. The *independence number* α of G is the maximum cardinality of an independent set of G. The independence number is one of the most fundamental and well-studied graph parameters [6]. In view of its computational hardness [5] various bounds on the independence number have been proposed. The following classical bound holds for every graph G and is due to Caro and Wei [1, 7]

$$\alpha \quad \geq \quad \sum_{u \in V} \frac{1}{d(u)+1}. \tag{1}$$

Since the only graphs for which (1) is best-possible are the disjoint unions of cliques, additional structural assumptions excluding these graphs allow improvements of (1). A natural candidate for such assumptions is connectivity.
For connected graphs, Harant and Rautenbach proved [2] (cf. also [3] and [4])

Theorem 1 *If G is a connected graph, then there exist a positive integer $k \in \mathbb{N}$ and a function $\phi : V \to \mathbb{N}_0$ with non-negative integer values such that $\phi(u) \leq d(u)$ for all $u \in V$,*

$$\alpha \geq k \geq \sum_{u \in V} \frac{1}{d(u)+1-\phi(u)}, \tag{2}$$

and

$$\sum_{u \in V} \phi(u) \geq 2(k-1). \tag{3}$$

Note that Theorem 1 is best-possible for the connected graphs which arise by adding bridges to disjoint unions of cliques, i.e. it is best-possible for the intuitively most natural candidate of a connected graph with small independence number. In [3], a weaker version of Theorem 1 is proved. This result is obtained from Theorem 1 by replacing the inequality (3) by $\sum_{u\in V}\phi(u)\geq k-1$.

For an integer l with $0\leq l\leq 2m$ let $f(l)=\min\sum_{u\in V}\frac{1}{d(u)+1-\phi(u)}$, where the minimum is taken over all integers $0\leq\phi(u)\leq d(u)$ with $\sum_{u\in V}\phi(u)=l$.

Obviously, f is strictly increasing. With this function f, it follows the existence of positive integers k_1 and k_2 such that $\alpha\geq k_1+1\geq f(k_1)$ (put $k_1=k-1$ and use the result in [3]) and $\alpha\geq\frac{k_2}{2}+1\geq f(k_2)$ (with $k_2=2(k-1)$ and Theorem 1). After extending f to real arguments, in [4], it is proved that the function $l+1-f(l)$ is continuous and strictly increasing and that k_1 is at least the unique zero k_0 of this function. Finally, $\alpha\geq k_0+1$ is the main result in [4].

Here we will show that the continuous function $\frac{l}{2}+1-f(l)$ is also strictly increasing. If we assume that $\sum_{u\in V}\frac{1}{d(u)+1}\geq 2$ for the graph in question then $f(2)>f(0)=\sum_{u\in V}\frac{1}{d(u)+1}\geq 2$. It will be proved that there is a unique solution l_0 of the equation $\frac{l}{2}+1=f(l)$ and, because $\frac{l}{2}+1-f(l)$ is strictly increasing and $\frac{2}{2}+1-f(2)<0$, it follows $l_0>2$. Consequently, $\frac{l_0}{2}+1=f(l_0)>f(\frac{l_0}{2}+1)$, since f is strictly increasing, hence, $\frac{l_0}{2}>k_0$.

The inequality $\alpha\geq\frac{l_0}{2}+1$ is the content of the following Theorem 2.

In case $\sum_{u\in V}\frac{1}{d(u)+1}\geq 2$, Remark 2 gives a lower bound on the improvement

$(\frac{l_0}{2}+1)-(k_0+1)=f(l_0)-f(k_0)$.

Theorem 2 *Let G be a finite, simple, connected, and non-complete graph on $n\geq 3$ vertices of size $m\geq n$. Moreover, let $\alpha\leq\frac{n}{2}$ be the independence number, Δ be the maximum degree of G, n_j be the number of vertices of degree j in G, and*

$$x(j)=\frac{j(j+1)}{j(j+1)-2}\left((\frac{2}{j+1}-(\Delta-j))n_\Delta+...+(\frac{2}{j+1}-1)n_{j+1}+\frac{2n_j}{j+1}+...+\frac{2n_1}{2}-2\right)$$

for $j\in\{2,...,\Delta\}$.

Then
(i) there is a unique $j_0\in\{2,...,\Delta\}$ such that $0\leq x(j_0)<n_\Delta+...+n_{j_0}$ and

(ii)

$$\alpha\geq(\sum_{u\in V}\frac{1}{d(u)+1})+\frac{n_\Delta}{\Delta(\Delta+1)}+\frac{n_\Delta+n_{\Delta-1}}{(\Delta-1)\Delta}+...+\frac{n_\Delta+...+n_{j_0+1}}{(j_0+2)(j_0+1)}+\frac{x(j_0)}{(j_0+1)j_0}$$

$$=\frac{x(j_0)+n_{j_0+1}+2n_{j_0+2}+...+(\Delta-j_0)n_\Delta}{2}+1.$$

2 Proof of Theorem 2

In the sequel let k be the lower bound on α of Theorem 1.

By Theorem 1, it follows

Lemma 1 $k\geq f(2(k-1))$.

For a finite family F of integers let $max(F)$ be a maximum member of F. Note that a member of a family may occur more than once. If for instance $F = \{1, 2, 2\}$ then $(F \setminus \{max(F)\}) \cup \{max(F) - 1\} = \{1, 1, 2\}$. The following Lemma 2, Lemma 3, and Lemma 4 are proved in [3] and [4].

Lemma 2 *Given an integer l with $0 \leq l \leq 2m$, the following algorithm calculates $f(l)$:*
Input: The family $F = \{d(u) \mid u \in V\}$.
$j := 0,$
while $j < l$ do begin $F := (F \setminus \{max(F)\}) \cup \{max(F) - 1\}$; $j := j + 1$ end
Output: $f(l) = \sum\limits_{m \in F} \frac{1}{m+1}$.

Lemma 3 is a consequence of Lemma 2.

Lemma 3 *Given an integer $0 \leq l \leq 2m$,*
(i) there are unique integers j and x with $j \in \{1, ..., \Delta\}$ and $x \in \{0, ..., n_\Delta + ... + n_j - 1\}$ such that
$l = n_\Delta + (n_\Delta + n_{\Delta-1}) + ... + (n_\Delta + n_{\Delta-1} + ... + n_{j+1}) + x$
$= x + n_{j+1} + 2n_{j+2} + ... + (\Delta - j)n_\Delta$
and
(ii) $f(l) = (n_\Delta + ... + n_j - x)\frac{1}{j+1} + \frac{x}{j} + \frac{n_{j-1}}{j} + ... + \frac{n_1}{2}$
$= (n_\Delta + ... + n_j)\frac{1}{j+1} + \frac{x}{j(j+1)} + \frac{n_{j-1}}{j} + ... + \frac{n_1}{2}$.

By Lemma 3, it follows

Lemma 4 *If $l = x + n_{j+1} + 2n_{j+2} + ... + (\Delta - j)n_\Delta$ with $j \in \{1, ..., \Delta\}$ and $x \in \{0, ..., n_\Delta + ... + n_j - 1\}$ than $f(l + 1) - f(l) = \frac{1}{j(j+1)}$.*

Using Lemma 3, the calculation of $f(l)$ is possible without taking a minimum and without using the algorithm above. We will now define the function f for real $l \in [1, m)$. For given $j \in \{1, ..., \Delta\}$ and a real number x with $0 \leq x < n_\Delta + ... + n_j$ let the real numbers l and $f(l)$ (implicitly) be defined as $l = x + n_{j+1} + 2n_{j+2} + ... + (\Delta - j)n_\Delta$ and $f(l) = (n_\Delta + ... + n_j)\frac{1}{j+1} + \frac{x}{j(j+1)} + \frac{n_{j-1}}{j} + ... + \frac{n_1}{2}$.
We will prove Lemma 5.

Lemma 5 *The function g with $g(l) = \frac{l}{2} + 1 - f(l)$ is continuous and strictly increasing on $[1, n)$.*

Proof. Consider $l \in [1, n)$. Then there are $j \in \{1, ..., \Delta\}$ and x with $0 \leq x < n_\Delta + ... + n_j$ such that $l = x + n_{j+1} + 2n_{j+2} + ... + (\Delta - j)n_\Delta$.
If $j = 1$ then $n > l \geq n_2 + 2n_3 + ... + (\Delta - 1)n_\Delta = 2m - n$, a contradiction to $n \leq m$. Hence, $j \geq 2$, and l belongs to the interval
$I(j) = [n_{j+1} + 2n_{j+2} + ... + (\Delta - j)n_\Delta, n_j + 2n_{j+1} + ... + (\Delta - j + 1)n_\Delta)$.
By Lemma 3, $g(l + \epsilon) - g(l) = \epsilon(\frac{1}{2} - \frac{1}{j(j+1)})$ and, consequently, $g(l)$ is continuous and, because $j \geq 2$, strictly increasing on $I(j)$.
Note that $I(j) \cap I(j') = \emptyset$ if $j \neq j'$ and that $I(2) \cup ... \cup I(\Delta) = [1, 2m - n) \supseteq [1, n)$.
It is easy to see that g is also continuous in $l = n_{j+1} + 2n_{j+2} + ... + (\Delta - j)n_\Delta$ for $j \in \{3, ..., \Delta - 1\}$. □

Since the classical bound due to Caro and Wei is tight only for complete graphs, it follows $g(0) = 1 - \sum\limits_{u \in V} \frac{1}{d(u)+1} < 0$, and, by Lemma 1, $g(2(k - 1)) \geq 0$. Using Lemma 5, there is a unique zero $l_0 = x(j_0) + n_{j_0+1} + 2n_{j_0+2} + ... + (\Delta - j_0)n_\Delta$ of g with $1 < l_0 \leq 2(k - 1) \leq 2(\alpha - 1) < n$ and $0 \leq x(j_0) < n_\Delta + ... + n_{j_0}$. It follows Lemma 6.

Lemma 6 $\alpha \geq k \geq \frac{l_0}{2} + 1$, *where $l_0 \in (0, n]$ is the unique solution of $\frac{l}{2} + 1 = f(l)$.*

Considering the equation $\frac{l_0}{2}+1=f(l_0)$, i.e.
$2+x(j_0)+n_{j_0+1}+2n_{j_0+2}+...+(\Delta-j_0)n_\Delta = 2((n_\Delta+...+n_{j_0})\frac{1}{j_0+1}+\frac{x(j_0)}{j_0(j_0+1)}+\frac{n_{j_0-1}}{j_0}+...+\frac{n_1}{2})$
it follows
$x_0=\frac{j_0(j_0+1)}{j_0(j_0+1)-2}\Bigg((\frac{2}{j_0+1}-(\Delta-j_0))n_\Delta+...+(\frac{2}{j_0+1}-1)n_{j_0+1}+\frac{2n_{j_0}}{j_0+1}+...+\frac{2n_1}{2}-2\Bigg).$
We obtain Lemma 7.

Lemma 7 *If* $j\in\{2,...,\Delta\}$ *and* $l=x+n_{j+1}+2n_{j+2}+...+(\Delta-j)n_\Delta$ *with* $0\le x<n_\Delta+...+n_j$, *then* $\frac{l}{2}+1=f(l)$ *if and only if*

$$x=\frac{j(j+1)}{j(j+1)-2}\Bigg((\frac{2}{j+1}-(\Delta-j))n_\Delta+...+(\frac{2}{j+1}-1)n_{j+1}+\frac{2n_j}{j+1}+...+\frac{2n_1}{2}-2\Bigg).$$

Now we complete the proof of Theorem 2. By Lemma 4 and Lemma 6,
$\alpha\ge k\ge f(l_0)=f(0)+(f(1)-f(0))+...+(f(\lfloor l_0\rfloor)-f(\lfloor l_0\rfloor-1))+(f(l_0)-f(\lfloor l_0\rfloor))$
$=(\sum\limits_{u\in V}\frac{1}{d(u)+1})+\frac{n_\Delta}{\Delta(\Delta+1)}+\frac{n_\Delta+n_{\Delta-1}}{(\Delta-1)\Delta}+...+\frac{n_\Delta+...+n_{j_0+1}}{(j_0+2)(j_0+1)}+\frac{x(j_0)}{(j_0+1)j_0}$ because
$l_0=x(j_0)+n_{j_0+1}+2n_{j_0+2}+...+(\Delta-j_0)n_\Delta$
$=n_\Delta+(n_\Delta+n_{\Delta-1})+...+(n_\Delta+n_{\Delta-1}+...+n_{j_0+1})+x(j_0)$.
With $f(l_0)=\frac{l_0}{2}+1=\frac{x(j_0)+n_{j_0+1}+2n_{j_0+2}+...+(\Delta-j_0)n_\Delta}{2}+1$ Theorem 2 is proved. □

3 Remarks

The following Remark 1 is proved in the introduction.

Remark 1 *If* $\sum\limits_{u\in V}\frac{1}{d(u)+1}\ge 2$ *then* $\frac{l_0}{2}>k_0$.

Remark 2 compares the lower bound $\frac{l_0}{2}+1$ on α in Theorem 2 to the lower bound k_0+1 on α in the main result in [4].

Remark 2 *If* $\sum\limits_{u\in V}\frac{1}{d(u)+1}\ge 2$ *and*
$k_0=n_\Delta+(n_\Delta+n_{\Delta-1})+...+(n_\Delta+n_{\Delta-1}+...+n_{j+1})+x$
with $0\le x<n_\Delta+...+n_j$ *then* $\frac{l_0}{2}-k_0\ge\frac{k_0}{j(j+1)}$.

Proof. Remark 1 implies $\frac{l_0}{2}-k_0=f(l_0)-f(k_0)>f(2k_0)-f(k_0)$.
According to Lemma 2, the family F contains the member 1, the member 2 ,..., and the member Δ exactly n_1 times, n_2 times,..., n_Δ times, respectively.
Therefore, let the output $f(l)$ of the algorithm in Lemma 2 be denoted by $f_{n_1,...,n_\Delta}(l)$.
With this notation, for example $f_{n_1,...,n_\Delta}(1)=f_{n_1,...,n_{\Delta-1}+1,n_\Delta-1}(0)$.
Using Lemma 3 *(ii)*, it follows $f_{n_1,...,n_\Delta}(k_0)=f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x}(0)$ and
$f_{n_1,...,n_\Delta}(2k_0)=f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x}(k_0)$.
Consequently,
$f_{n_1,...,n_\Delta}(2k_0))-f_{n_1,...,n_\Delta}(k_0)=f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x}(k_0)-f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x}(0)$
$=(f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x)}(k_0)-f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x)}(k_0-1))$
$+(f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x)}(k_0-1)-f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x)}(k_0-2))+...$
$+(f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x)}(1)-f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x)}(0))$.
Note that the expressions $f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x)}(s)-f_{n_1,...,n_{j-2},n_{j-1}+x,n_\Delta+...+n_j-x)}(s-1)$ equal fractions of type $\frac{1}{a(a+1)}$ (see Lemma 3 and Lemma 4) with $a\le j$ for $s=1,...,k_0$.
Thus, $f_{n_1,...,n_\Delta}(2k_0)-f_{n_1,...,n_\Delta}(k_0)\ge\frac{k_0}{j(j+1)}$. □

For integers $r \geq 2$ and $s \geq 2$, consider the graph $G_{r,s}$ obtained from s copies of the clique K_r on r vertices and adding $s-1$ mutually independent edges between these cliques such that $G_{r,s}$ is connected. It follows $\Delta = r$, $n_j = 0$ for $j < r-1$, $n_{r-1} = sr - 2(s-1)$, $n_r = 2(s-1)$, and $\alpha = s$ for $G_{r,s}$. Using Theorem 2, we obtain

$$x(r-1) = \frac{(r-1)r}{(r-1)r-2}\left((\frac{2}{r}-1)n_r + \frac{2n_{r-1}}{r} - 2\right) = \frac{(r-1)r}{(r-1)r-2}\left((\frac{2}{r}-1)2(s-1) + \frac{2sr-4(s-1)}{r} - 2\right) = 0 \ .$$

Hence, $j_0 = r-1$,
$\frac{x(j_0)+n_{j_0+1}+2n_{j_0+2}+\ldots+(\Delta-j_0)n_\Delta}{2} + 1 = \frac{n_r}{2} + 1 = s = \alpha$, and Remark 3 follows.

Remark 3 *There are infinitely many graphs G such that the lower bound on α of Theorem 2 is tight.*

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